MINIMAL NORMAL COMPACTIFICATIONS OF $\mathbf{C}^2$

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§1. We want to classify compactifications of $\mathbf{C}^2$. A *compactification* of $\mathbf{C}^2$ is a nonsingular compact complex manifold M of complex dimension 2 which contains a nonempty nowhere dense closed analytic subset A such that $M - A$ is biholomorphic to $\mathbf{C}^2$. In the next section we show that A is a compact connected pure one-dimensional analytic set, and hence is a finite union of irreducible curves. By blowing up certain points of A we may assume that A has the following properties.

(1) $A = \bigcup_{i=1}^{k} \Gamma_i$, where Γ_i is a non-singular connected compact algebraic curve.

(2) Γ_i intersects Γ_j normally (if at all), and in at most one point.

(3) $\Gamma_i \cap \Gamma_j \cap \Gamma_l = \varnothing$ for any three distinct indices.

(4) If Γ_i is a non-singular rational curve and if the self-intersection $(\Gamma_i^2) = -1$, then blowing Γ_i down will give an A' which violates one of (1), (2), or (3).

We call such a compactification a *minimal normal compactification* (of $\mathbf{C}^2$). In this paper we shall find all minimal normal compactifications of $\mathbf{C}^2$. As a corollary of our constructions we find that all compactifications of $\mathbf{C}^2$ are rational. This result has been obtained by Kodaira [*4*] by other techniques. Our results depend heavily on results and techniques developed by Van de Ven [*10*] and Ramanujam [*7*]. In the last section we shall collect a list of open problems. These results were announced in [*6*].

§2. Let the notation be as in §1.

Lemma 1. *A is a connected pure one-dimensional set, hence a finite union of irreducible curves.*

Proof. It is clear that $\dim_{\mathbf{C}} A_x \leqq 1$ where A_x is the germ of A at $x \in A$. Suppose that A has two connected components $A = B \cup C$. By considering disjoint neighborhoods of B and C which retract down to B and C, we see that $\mathbf{C}^2$ is not connected at ∞. But this is a contradiction. If $\dim_{\mathbf{C}} A = 0$, then A is a point and M is homeomorphic to S^4. But S^4 has no complex structure (Borel and Serre [*1*]). Thus $\dim_{\mathbf{C}} A = 1$, and since A is compact we are done. (This can also be proved by using Hartogs' Theorem.) Q.E.D.

By the resolution of singularities for an algebraic curve and by the theorem of Castelnuovo-Enriques for a compact surface (see Kodaira [2]), we may assume that A satisfies (1) through (4) of §1. Let $A = \cup \Gamma_i$ as in §1.

Lemma 2. *$A = \cup \Gamma_i$ satisfies the further conditions*

(5) Each Γ_i is a nonsingular rational curve ($\Gamma_i = \mathbf{P}^1(\mathbf{C})$).

(6) A contains no cycles, i.e., there is no sequence $\Gamma_{i_1}, \cdots, \Gamma_{i_a}$ such that $\Gamma_{i_k} \cap \Gamma_{i_{k+1}} \neq \varnothing$ for $k = 1, \cdots, a-1$ and $\Gamma_{i_a} \cap \Gamma_{i_1} \neq \varnothing$.

Proof. (Van de Ven, Ramanujam). We give the proof in three parts.

(a) We first show $H^1(A) = 0$. By Poincaré duality we have

$$H_k(M-A) = H_C^{4-k}(M-A), \quad 0 \leqq k \leqq 4$$

where H_C^q denotes the qth cohomology group with compact support. Consider the exact cohomology sequence

$$\cdots \to H_C^k(M-A) \to H^k(M) \to H^k(A) \to H_C^{k+1}(M-A) \to \cdots$$

Now $M-A = \mathbf{C}^2$ so $H^k(M) = H^k(A)$ for $0 \leqq k \leqq 2$ since $H_C^k(\mathbf{C}^2) = 0$ for $0 \leqq k \leqq 3$. So $H^1(M) = H^1(A)$. By duality $H^3(M) = H_1(M)$. Since $\dim_{\mathbf{C}}(A) = 1$, $H^3(A) = 0$. The above exact sequence gives $H^3(M) = 0$. But $H_1(M) = 0$ implies that $0 = H^1(M) = H^1(A)$.

(b) We claim $H^1(\Gamma_i) = 0$ and hence (5) is true. Let $C = \Gamma_i$ and $B = \bigcup_{j \neq i} \Gamma_j$. Then $C \cap B$ is a finite collection of points. The Mayer-Vietoris sequence gives

$$0 = H^1(A) \to H^1(B) \oplus H^1(C) \to H^1(B \cap C) = 0$$

Thus $H^1(C) = 0$.

(c) We prove (6). In this case let $B = \Gamma_{i_1}$, $C = \bigcup_{j=2}^{a} \Gamma_{i_j}$. Mayer-Vietoris again gives

$$0 \to H^0(B \cup C) \to H^0(B) \oplus H^0(C) \to H^0(B \cap C) \to H^1(B \cup C).$$

This becomes

$$0 \to \mathbf{Z} \to \mathbf{Z}^2 \to \mathbf{Z}^r \to H^1(B \cup C)$$

where r is the number of points in $B \cap C$. Since $r \geqq 2$, $H^1(B \cup C) \neq 0$. Let D be the union of the rest of the curves in A. Then Mayer-Vietoris gives

$$0 = H^1(A) \to H^1(D) \oplus H^1(B \cup C) \to H^1(D \cap (B \cup C)) = 0.$$

This yields the contradiction $H^1(B \cup C) = 0$. Thus there are no cycles. Q.E.D.

Before we state any more properties of A we need to quote the following result of Van de Ven [*10*].

Theorem α. *M is algebraic.*

Each Γ_i is a divisor on M. The vector space (over $\mathbf{R}$) with the Γ_i as basis has on it a quadratic form defined by intersection multiplicity. Let b^+ be the number of positive eigenvalues of the intersection form on this space.

Lemma 3. *We have the following additional properties of M and A.*

(7) $b^+ = 1$.

(8) $H^1(M,\mathcal{O}) = 0$ where $\mathcal{O}$ is the sheaf of germs of holomorphic functions on M.

Proof. According to a theorem of Kodaira (see Kodaira [*3*], p. 755) $b^+ = 2h^{2,0} + 1$, where $h^{2,0} = \dim H^0(M,\Omega^2) = \dim H^2(M,\mathcal{O})$ and Ω^2 is the sheaf of germs of holomorphic 2-forms. We want to show that $h^{2,0} = 0$. Generally we define $h^{p,q} = \dim H^q(M,\Omega^p)$, where Ω^p is the sheaf of germs of holomorphic p-forms. Since M is Kähler we have

$$\begin{aligned} b_2 &= h^{2,0} + h^{1,1} + h^{0,2} \\ &= 2h^{2,0} + h^{1,1} \end{aligned}$$

where b_n is the nth Betti number of M. Now we have proved that $H^2(M) \to H^2(A)$ is an isomorphism. But Mayer-Vietoris again shows that $H^2(A)$ is freely generated by the classes of the Γ_i and hence $H^2(M)$ is freely generated by the Poincaré duals of the Γ_i. But these are all of type $(1,1)$ hence $b_2 = h^{1,1}$. Thus $h^{2,0} = 0$. Since M is Kähler, $2h^{0,1} = b_1 = 0$. Q.E.D.

With each collection of curves A we associate a dual weighted graph $\gamma(A)$. We call this graph the *graph of the compactification.* Each vertex of the graph represents a nonsingular rational curve Γ_i. Adjacent to each vertex we write the self-intersection (Γ_i^2) of the curve Γ_i. These are called the *weights*. If two vertices are joined by a segment this means that the two rational curves they represent intersect. By a *branch point* we mean a vertex which is attached to at least 3 segments. A graph is *linear* if it has no cycles or branch points. The following theorem is the main result of Ramanujam [*7*].

Theorem β. *The graph of A is linear.*

This result is derived using only the assumptions that A is connected, that the pair (M,A) satisfies (*1*) through (*8*), and that the fundamental group of the boundary of a tubular neighborhood of A is trivial.

§3. Let the length $l(\gamma(A))$ of the graph $\gamma(A)$ be the number of vertices in the graph (with the notation of §§1, 2)

Lemma 4. *We have the following list of possible graphs of minimal normal compactifications of* $\mathbf{C}^2$:

$l(\gamma(A))$	graph	
1	$\circ\ 1$	
2	$\circ\ n$ — $\circ\ 0$	$n \neq -1$.
3	$\circ\ n$ — $\circ\ 0$ — $\circ\ -n-1$	$n > 0$

Proof. In case $l(\gamma(A)) = l = 1$, let p be the self intersection (A^2). The fundamental group Π_1^∞ at ∞ of $\mathbf{C}^2$ is trivial. Hence the fundamental group of the boundary of a tubular neighborhood of A is trivial. But according to Mumford [5], generally, the fundamental group of the boundary of a tubular neighborhood of A has generators $\{e_1, \cdots, e_k\}$ where $A = \bigcup_{i=1}^k \Gamma_i$ subject to the following relations:

(a) $$e_i e_j = e_j e_i, \quad \text{if} \quad (\Gamma_i \Gamma_j) = s_{ij} \neq 0$$

(b) $$e_1^{s_{1q}} \cdot e_2^{s_{2q}} \cdots e_k^{s_{kq}} = 1, \quad \text{for} \quad 1 \leqq q \leqq k$$

For $l = 1$ we have one generator e and $e^p = 1$. Thus $\Pi_1^\infty = \mathbf{Z}/p\mathbf{Z} \neq 1$ unless $p = \pm 1$. But $p = -1$ is not allowed by (4), §1. Thus the first statement is true.

For the second statement we have generators e_1, e_2 with relations

$$e_1 e_2 = e_2 e_1$$

$$e_1 \cdot e_2^{n_2} = 1$$

$$e_1^{n_1} \cdot e_2 = 1$$

where $n_i = (\Gamma_i^2)$. Thus Π_1^∞ has generator e_2 and relation $e_2^{n_1 n_2 - 1} = 1$. If $\Pi_1^\infty = 1$, then $n_1 n_2 - 1 = \pm 1$. If $n_1 n_2 = 2$, then we have (say) $n_1 = 1$, $n_2 = 2$, or $n_1 = -1$, $n_2 = -2$. The case $n_1 = -1$ is ruled out by (4), §1. The case $n_1 = 1$, $n_2 = 2$ is ruled out by (7), §2. Thus $n_1 n_2 = 0$, and the second statement is true.

To prove the third statement we refer to another result of Ramanujam [7], pp. 78–79.

Lemma γ. *Let* $A, \gamma(A)$ *be as in sections* 1 *and* 2. *Then*

(i) The graph $\gamma(A)$ has at least one non-negative weight and at most two non-negative weights.

(ii) If $\gamma(A)$ has exactly one non-negative weight, then $l(\gamma(A)) = 1$ or 2; if $l(\gamma(A)) = 1$ it is 1, and if $l(\gamma(A)) = 2$ it is 0.

(iii) If $\gamma(A)$ has exactly 2 non-negative weights, then the vertices they correspond to must be linked and one of them must have weight 0.

There are only two essentially different graphs with $l(\gamma(A)) = 3$. According to Lemma γ they take the forms

$$\overset{0}{\circ}\!-\!\overset{n}{\circ}\!-\!\overset{k}{\circ}\,, \qquad \overset{n}{\circ}\!-\!\overset{0}{\circ}\!-\!\overset{k}{\circ}$$

where $n \geqq 0$. Consider the first type. Blow up the intersection point of the curves corresponding to the vertices with weight 0 and n. Then blow down the proper transform of the curve corresponding to the vertex with weight 0. Repeat this process $n+1$ times. We then get a new manifold M' and collection of rational curves A' with graph

$$\overset{0}{\circ}\!-\!\!-\!\overset{-1}{\circ}\!-\!\!-\!\overset{k}{\circ}$$

such that $M' - A' = \mathbf{C}^2$. Contract the curve with self-intersection -1 to get a compactification of $\mathbf{C}^2$ by a collection of curves with graph

$$\overset{1}{\circ}\!-\!\!-\!\overset{k+1}{\circ} \quad .$$

Now if $k+1 = -1$, we blow down the curve with self-intersection $k+1$ to get a compactification of $\mathbf{C}^2$ by a rational curve with self-intersection 2 contradicting the first part of this lemma. Thus $k = -1$ by the second part of this lemma. But this contradicts (4), §1. Thus the only possibility is

$$\overset{n}{\circ}\!-\!\!-\!\overset{0}{\circ}\!-\!\!-\!\overset{k}{\circ}\,.$$

Blow up the point of intersection of the curve with self-intersection n with the curve with self-intersection 0, and blow down the proper transform of the curve with self-intersection 0. Repeat this $n+1$ times to get a compactification of $\mathbf{C}^2$ bv a collection of rational curves with graph

$$\overset{-1}{\circ}\!-\!\!-\!\overset{0}{\circ}\!-\!\!-\!\overset{k+n+1}{\circ} \quad .$$

Then we can get a compactification of $\mathbf{C}^2$ by curves with graph

$$\overset{1}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{k+n+1}{\circ} .$$

Either $k+n+1 = -1$, which leads to a contradiction, or $k = -n-1$. Since $k = -1$ is not allowed, $n > 0$. Q.E.D.

Lemma 5. *If $l(\gamma(A)) = 1$, then M is $\mathbf{P}^2$ and $A = \Gamma$ is a line in $\mathbf{P}^2$.* (See also Remmert and Van de Ven [8]).

Proof. Let $F = [\Gamma]$ denote the bundle associated to the divisor Γ. Then we have the following exact sequence

$$\begin{array}{ccccccc} 0 \to H^0(M,\mathcal{O}) & \to & H^0(M,\mathcal{O}(F)) & \to & H^0(M,\mathcal{O}(F|_\Gamma)) & \to & H^1(M,\mathcal{O}) \\ \| & & & & \| & & \| \\ \mathbf{C} & & & & \mathbf{C}^2 & & 0 \end{array}$$

Thus $H^0(M,\mathcal{O}(F)) = \mathbf{C}^3$. Let $\{U_i\}$ be an open covering of M and let $f_i = 0$ be a local defining equation for $\Gamma \cap U_i$. Then $\phi_1 = \{f_i\}$ is a section of F. We can find two other sections ϕ_2, ϕ_3 so that $\{\phi_2|_\Gamma, \phi_3|_\Gamma\}$ gives a basis for $H^0(M,\mathcal{O}(F|_\Gamma))$. Let $p \in M$ be such that $\phi_1(p) = \phi_2(p) = \phi_3(p) = 0$. Then $p \in \Gamma$ and $\phi_2(p) = \phi_3(p) = 0$. But one can find a linear form $l = a\phi_2 + b\phi_3$ on Γ with $l(p) \neq 0$. Thus there is no such p, and $p \xrightarrow{\phi} [\phi_1(p), \phi_2(p), \phi_3(p)] \in \mathbf{P}^2$ defines a holomorphic map into $\mathbf{P}^2$.

We claim ϕ is 1–1 (and thus onto). Let $q \in \mathbf{P}^2$, and let $L_1: \sum_{i=1}^3 \alpha_i\zeta_i = 0$, L_2: $\sum_{i=1}^3 \beta_i\zeta_i = 0$ define two distinct lines which intersect at q. Then D_1: $\sum_{i=1}^3 \alpha_i\phi_i(p) = 0$, D_2: $\sum_{i=1}^3 \beta_i\phi_i(p) = 0$ define two divisors on M and $\phi^{-1}(q) = D_1 \cap D_2$. Since $(\Gamma^2) = 1$, $(D_1 \cdot D_2) = 1$. Suppose D_1 and D_2 have a common component D. Then $(D^2) < 0$. But $c(D) = nc(\Gamma)$ where c denotes Poincaré dual. Thus $(D^2) = n^2 > 0$. Hence D_1 and D_2 have no common component. Thus $\phi^{-1}(q) = D_1 \cap D_2$ is one point. Thus ϕ is an isomorphism. Q.E.D.

Lemma 6. *A weighted graph of the form*

$$\overset{k_r}{\circ}\!\!-\!\!\!-\!\!\overset{k_{r-1}}{\circ}\!\!-\!\!\cdots\!\!-\!\!\overset{k_1}{\circ}\!\!-\!\!\!-\!\!\overset{n}{\circ}\!\!-\!\!\!-\!\!\overset{0}{\circ}, \quad n > 0,\ r \geqq 1$$

cannot occur as the graph of a minimal normal compactification of $\mathbf{C}^2$.

Proof. If such a graph occurs, then proceeding as in Lemma 4 we see that

$$\overset{k_r}{\circ}\!\!-\!\!\cdots\!\!-\!\!\overset{k_1+1}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{1}{\circ}$$

is also the graph of a compactification. Then either $k_1+1=-1$ or $k_1+1=0$. But $k_1=-1$ is not allowed. Continuing, we find that $\circ\ r+1$ is a graph of a compactification, contradicting Lemma 4. Q.E.D.

Lemma 7. *If*

$$\overset{k_r}{\circ}-\cdots-\overset{k_1}{\circ}-\overset{n}{\circ}-\overset{0}{\circ}-\overset{l_1}{\circ}-\cdots-\overset{l_s}{\circ}, \quad n \geqq 0,\ r \geqq 0,\ s \geqq 1$$

is the graph of a minimal normal compactification of $\mathbf{C}^2$, *then* $l_1=-n-1$, *and* $n>0$.

Proof. If we have such a graph, then

$$\overset{k_r}{\circ}-\cdots-\overset{k_1+1}{\circ}-\overset{1}{\circ}-\overset{l_1+n+1}{\circ}-\cdots-\overset{l_s}{\circ}$$

is also the graph of a compactification. Now $k_r \leqq -2$, and it is posssible that $k_1=-2$. Then $k_1+1=-1$, and

$$\overset{k_r}{\circ}-\cdots-\overset{k_2+1}{\circ}-\overset{2}{\circ}-\overset{l_1+n+1}{\circ}-\cdots-\overset{l_s}{\circ}$$

is the graph of a compactification. Continuing this way we get a graph of a minimal normal compactification of the form

$$\overset{k'_t}{\circ}-\cdots-\overset{k'_1}{\circ}-\overset{m}{\circ}-\overset{l_1+n+1}{\circ}-\cdots-\overset{l_s}{\circ}$$

where $t \geqq 0$, $k'_i \leqq -2$, and $m>0$. By Lemma γ, $l_1+n+1=0$, since $l_i \leqq -2$. Q.E.D.

Lemma 8. *If*

$$\overset{k_s}{\circ}-\cdots-\overset{k_1}{\circ}-\overset{n}{\circ}-\overset{0}{\circ}-\overset{-n-1}{\circ} \qquad n>0$$

$$\overset{n}{\circ}-\overset{0}{\circ}-\overset{-n-1}{\circ}-\overset{l_1}{\circ}-\cdots-\overset{l_r}{\circ} \qquad n>0$$

are graphs of minimal normal compactifications of $\mathbf{C}^2$, *then*

$$k_s=k_{s-1}=\cdots=k_1=l_1=\cdots=l_r=-2.$$

Proof. In the case of the first graph we find that

$$\overset{k_s}{\circ}-\cdots-\overset{k_1+1}{\circ}-\overset{1}{\circ}-\overset{0}{\circ}$$

is also the graph of a compactification. By Lemma 6, $k_1 = -2$. Then

$$\overset{k_s}{\circ}-\cdots-\overset{k_2+1}{\circ}-\overset{2}{\circ}-\overset{0}{\circ}$$

is the graph of a compactification. By repeated applications of Lemma 6 we get $k_1 = \cdots = k_s = -2$. In the case of the second graph we find that

$$\overset{1}{\circ}-\overset{0}{\circ}-\overset{l_1}{\circ}-\cdots-\overset{l_r}{\circ}$$

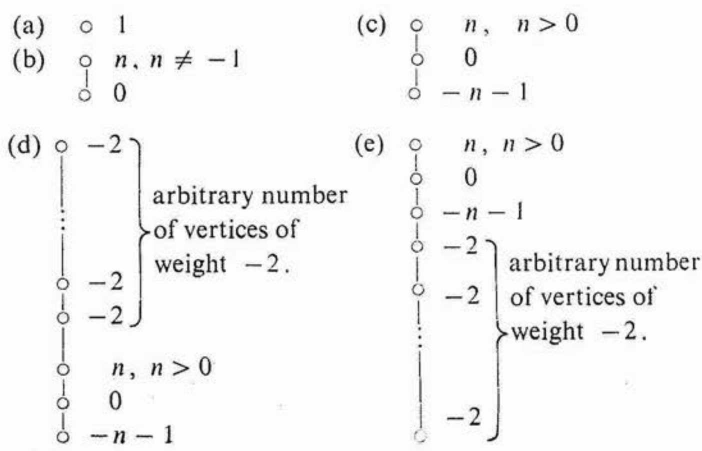
is the graph of a minimal normal compactification. By Lemma 7, $l_1 = -2$. Then

$$\overset{1}{\circ}-\overset{0}{\circ}-\overset{l_2}{\circ}-\cdots-\overset{l_r}{\circ}$$

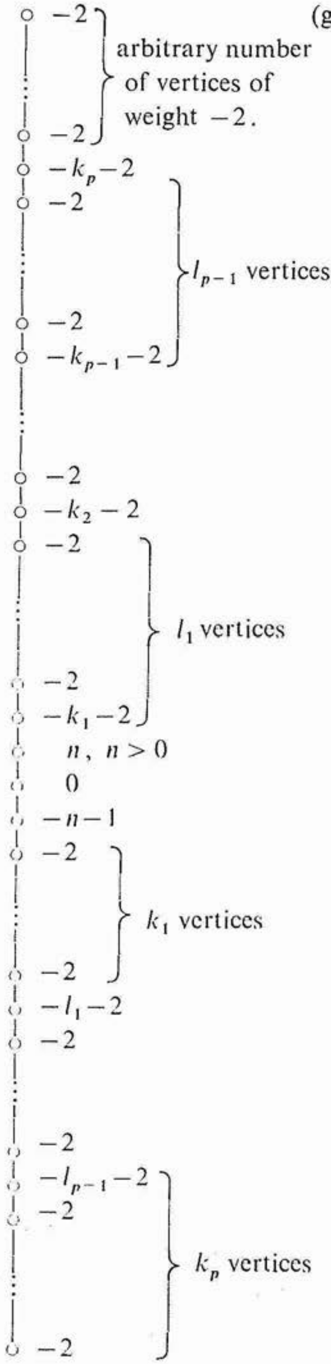
is the graph of a minimal normal compactification, and again $l_2 = -2$. Continuing we get $l_1 = \cdots = l_r = -2$. Q.E.D.

§4. In the rest of this paper we use the same notation as in the previous sections. Our main result is the following.

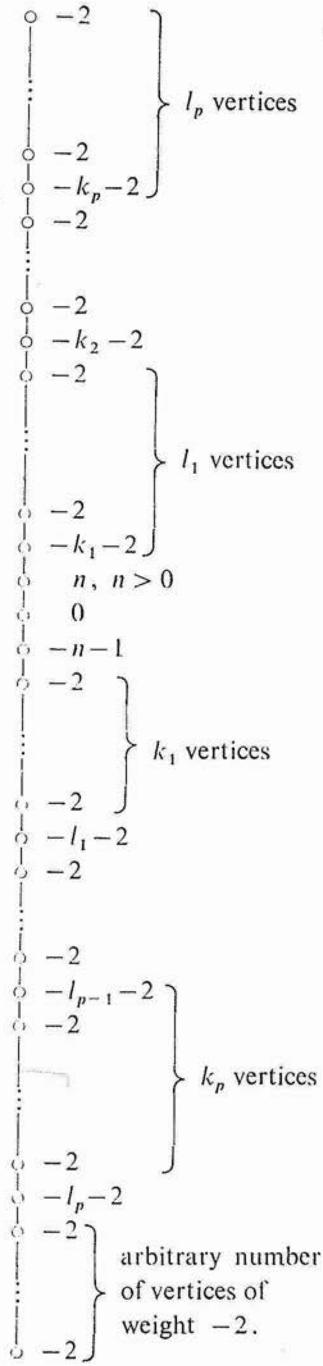
Theorem 9. *The following are the only possible graphs of minimal normal compactifications of* $\mathbf{C}^2$, *where we consider the graphs described in Lemmas* 4 *and* 8 *as special cases.* (*For the sake of completeness we reproduce them here.*) These graphs actually occur as graphs of compactifications.



(f)


-2
arbitrary number of vertices of weight -2.
-2
$-k_p-2$
-2
l_{p-1} vertices
-2
$-k_{p-1}-2$
-2
$-k_2-2$
-2
l_1 vertices
-2
$-k_1-2$
$n,\ n>0$
0
$-n-1$
-2
k_1 vertices
-2
$-l_1-2$
-2
-2
$-l_{p-1}-2$
-2
k_p vertices
-2

(g)


-2
l_p vertices
-2
$-k_p-2$
-2
-2
$-k_2-2$
-2
l_1 vertices
-2
$-k_1-2$
$n,\ n>0$
0
$-n-1$
-2
k_1 vertices
-2
$-l_1-2$
-2
-2
$-l_{p-1}-2$
-2
k_p vertices
-2
$-l_p-2$
-2
arbitrary number of vertices of weight -2.
-2

The graphs in (a), (b), (c), (d), and (e) are self-explanatory. In graph (f) the integers k_i, l_j are non-negative. We allow $k_1 = 0$, but we want $k_i > 0$ if $i > 1$. The integer l_1 is the number of vertices between the vertex with weight $-k_2 - 2$ and the vertex with weight n. The integer l_2 is one more than the number of vertices between the vertex with weight $-k_3 - 2$ and the vertex with weight $-k_2 - 2$, etc. In graph (f) if there are no vertices of weight less than -2 above the vertex of weight n, then there are no vertices below the vertex of weight $-n-1$. The case in which there are no vertices of weight less than -2 above the vertex of weight n and in which there appear some vertices below the vertex of weight $-n-1$ is covered by graph (g). One has similar remarks about graph (g) and how it is related to graph (f). Note that (d) is a special case of (f), (e) is a special case of (g), and (c) is a special case of both of them.

Proof. Let γ denote the graph of a minimal normal compactification of $\mathbf{C}^2$. We want to show that γ has the desired form. The proof goes by induction on $l(\gamma)$. If $l(\gamma) = 1, 2$, or 3 we are done by Lemma 4. So suppose $l(\gamma) > 3$. According to Lemma 7, γ has a subgraph of type

$$\begin{array}{ccccc} n & & 0 & & -n-1 \\ \circ & - & \circ & - & \circ \end{array} \quad , \quad n > 0.$$

Suppose the graph γ is

$$\begin{array}{ccccccccccccc} -a_r & & & -a_1 & & n & & 0 & & -n-1 & & -b_1 & & & -b_s \\ \circ & - & \cdots - & \circ & - & \circ & - & \circ & - & \circ & - & \circ & - & \cdots - & \circ \end{array} .$$

If $r = 0$ or $s = 0$, then by Lemma 8 we are done. So supppose $r > 0, s > 0$. By blowing up points and contracting curves as in Lemma 4, we find that the following graph is a graph of a compactification

$$\begin{array}{ccccccccccccc} -a_r & & & -a_1+1 & & 1 & & 0 & & -b_1 & & & -b_s \\ \circ & - & \cdots - & \circ & - & \circ & - & \circ & - & \circ & - & \cdots - & \circ \end{array} .$$

By Lemma γ and the fact that the given graph is minimal, $a_i \geqq 2$, $b_j \geqq 2$ for $1 \leqq i \leqq r$ and $1 \leqq j \leqq s$. If $a_1 \geqq 3$, then $-a_1 + 1 \leqq -2$. Thus by Lemma 7, $b_1 = 2$. Now by induction the graph has the desired form. If $a_1 = 2$, let $t - 1 \leqq r$ be the largest integer such that $a_1 = \cdots = a_{t-1} = 2$. Then the following graph is the graph of a minimal normal compactification

$$\begin{array}{ccccccccccccc} -a_r & & & -a_t+1 & & t & & 0 & & -b_1 & & & -b_s \\ \circ & - & \cdots - & \circ & - & \circ & - & \circ & - & \circ & - & \cdots - & \circ \end{array} .$$

Thus $-b_1 = -t - 1 = -(t-1) - 2$. In this picture we are assuming $t \leqq r$. Of course it may happen that $t = r + 1$. This graph is then

$$\overset{t}{\circ}\!-\!\!-\!\!-\!\overset{0}{\circ}\!-\!\!-\!\!-\!\overset{-t-1}{\circ}\!-\!\!-\!\!-\!\overset{-b_2}{\circ}\!-\!\cdots\!-\!\overset{-b_s}{\circ}$$

and $b_2 = \cdots = b_s = 2$, so that the original graph was of type (g). In the first case the induction assumption shows that our original graph was of type (f) or (g). This proves the first part of the theorem.

For the second part clearly $\mathbf{P}^2$ is a minimal normal compactification of $\mathbf{C}^2$ with graph (a). Let Γ be a line on $\mathbf{P}^2$. If we blow up a point of Γ we get a compactification of $\mathbf{C}^2$ with graph

$$\overset{-1}{\circ}\!-\!\!-\!\!-\!\overset{0}{\circ}\,.$$

If we blow up the point of intersection of these two curves and then blow down the proper transform of the curve with self-intersection 0, we get a compactification with graph

$$\overset{-2}{\circ}\!-\!\!-\!\!-\!\overset{0}{\circ}\;;$$

continuing this way we see that there are compactifications with graph

$$\overset{-n}{\circ}\!-\!\!-\!\!-\!\overset{0}{\circ}\qquad n > 0.$$

If, instead of blowing up the point of intersection, we blow up some other point on the curve with self-intersection 0, we may then blow down the proper transform of this curve to get a compactification with graph

$$\overset{0}{\circ}\!-\!\!-\!\!-\!\overset{0}{\circ}\,.$$

We can repeat this process to get a compactification with graph of the form

$$\overset{n}{\circ}\!-\!\!-\!\!-\!\overset{0}{\circ}\,,\qquad n \geqq 0.$$

If we take such a compactification with $n = 1$, we can blow up some point on the curve with self-intersection 1 different from the intersection of the two curves to get a compactification with graph

$$\overset{-1}{\circ}\!-\!\!-\!\!-\!\overset{0}{\circ}\!-\!\!-\!\!-\!\overset{0}{\circ}\,.$$

We then blow up the intersection of the curve of self-intersection 0 with the curve of self-intersection -1. Then blow down the proper transform of the curve with self-intersection 0. We get a compactification with graph

$$\overset{-2}{\circ}\!-\!\!-\!\overset{0}{\circ}\!-\!\!-\!\overset{1}{\circ}\,.$$

Repeating this we get a compactification with graph of type (c). Now we have shown in the proof of the first part of the theorem how to reduce a compactification to one with graph of type (c) (even type (b)). This procedure is reversible, so we see that there are compactifications of $\mathbf{C}^2$ which have any graph described in the first part of the theorem. Q.E.D.

Lemma 10. *Any minimal normal compactification of $\mathbf{C}^2$ is rational.*

Proof. From the proof of the theorem it follows that any compactification M of $\mathbf{C}^2$ can be obtained by some sequence of quadratic transformations and inverses of quadratic transformations from a compactification $\hat{M}$ of $\mathbf{C}^2$ by a non-singular rational curve Γ with $(\Gamma^2) = 1$. By Lemma 5 $\hat{M} = \mathbf{P}^2$. Q.E.D.

Theorem 11. *Any compactification of $\mathbf{C}^2$ is rational.*

Proof. This follows since any compactification is birationally equivalent to a minimal normal compactification. Q.E.D.

Theorem 12. *Any two minimal normal compactifications of $\mathbf{C}^2$ with the same graph γ are biholomorphically equivalent.*

Proof. We already know the result for $l(\gamma) = 1$ (Lemma 5). By the proof of Theorem 9, any compactification with graph $\overset{0}{\circ}\!-\!\overset{0}{\circ}$ can be obtained from $\mathbf{P}^2$ via compactifications with graphs as follows

$$\overset{1}{\circ} \longrightarrow \overset{0}{\circ}\!-\!\!-\!\overset{-1}{\circ} \longrightarrow \overset{-1}{\circ}\!-\!\!-\!\overset{-1}{\circ}\!-\!\!-\!\overset{-1}{\circ} \longrightarrow \overset{0}{\circ}\!-\!\!-\!\overset{0}{\circ} \tag{1}$$

We can find an automorphism of $\mathbf{P}^2$ which sends any line and two given points on that line into any other line and two given points on that line. Hence it follows that any two surfaces obtained from $\mathbf{P}^2$ by blowing up points and blowing down curves in the manner described by (1) are isomorphic. In fact if $M - A = \mathbf{C}^2 = M' - A'$ where $A = \Gamma_1 \cup \Gamma_2$, $A' = \Gamma_1' \cup \Gamma_2'$, and if A and A' both have graph $\overset{0}{\circ}\!-\!\!-\!\overset{0}{\circ}$, then M and M' are isomorphic in such a way that Γ_i is mapped to Γ_i' and $\Gamma_1 \cap \Gamma_2$ is mapped to $\Gamma'_1 \cap \Gamma'_2$. Thus any compactification with graph $\overset{0}{\circ}\!-\!\!-\!\overset{0}{\circ}$ is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$.

Now consider any compactification with graph $\overset{n}{\circ}\!-\!\!-\!\overset{0}{\circ}$ with $n < 0$. This compactification can be obtained from a compactification with graph $\overset{0}{\circ}\!-\!\!-\!\overset{0}{\circ}$ $(= \mathbf{P}^1 \times \mathbf{P}^1)$ via the sequence described in (2).

$$(2)\quad \overset{0}{\circ}-\overset{0}{\circ}\longrightarrow\overset{-1}{\circ}-\overset{-1}{\circ}-\overset{-1}{\circ}\longrightarrow\overset{-1}{\circ}-\overset{0}{\circ}\longrightarrow\overset{-2}{\circ}-\overset{-1}{\circ}-\overset{-1}{\circ}\longrightarrow\overset{-2}{\circ}-\overset{0}{\circ}\rightarrow\cdots \text{ etc.}$$

Thus all such compactifications are isomorphic, and in a way that preserves the curves of A and sends intersection points into intersection points.

Next consider two compactifications with graph $\overset{n}{\circ}\!-\!\!-\!\overset{0}{\circ}$ where $n > 0$. We show that they are isomorphic in a way that induces an isomorphism of the compactifying curves. We see that any compactification with graph $\overset{1}{\circ}\!-\!\!-\!\overset{0}{\circ}$ is produced from $\mathbf{P}^1 \times \mathbf{P}^1$ by a sequence of transformations described in (3).

$$(3)\quad \overset{0}{\circ}-\overset{0}{\circ}\longrightarrow\overset{0}{\circ}-\overset{-1}{\circ}-\overset{-1}{\circ}\longrightarrow\overset{1}{\circ}-\overset{0}{\circ}\;.$$

Since there is an automorphism of $\mathbf{P}^1 \times \mathbf{P}^1$ fixing $\mathbf{P}^1 \times \{0\}$ and sending $(0, a)$ to $(0, b)$ for $a, b \neq 0$ $(\mathbf{P}^1 = \mathbf{C} \cup \{\infty\})$, we see that any two manifolds produced by a sequence of operations described in (3) are isomorphic in the required way.

For the case $\overset{n}{\circ}\!-\!\!-\!\overset{0}{\circ}$ we proceed by induction on n. Any compactification with graph $\overset{n}{\circ}\!-\!\!-\!\overset{0}{\circ}$ is obtained from one with graph $\overset{n-1}{\circ}\!-\!\!-\!\overset{0}{\circ}$ via the following sequence.

$$(4)\quad \overset{n-1}{\circ}-\overset{0}{\circ}\longrightarrow\overset{n-1}{\circ}-\overset{-1}{\circ}-\overset{-1}{\circ}\longrightarrow\overset{n}{0}-\overset{0}{0}$$

Thus, the compactifications $\overset{n}{\circ}\!-\!\!-\!\overset{0}{\circ}$ are $\mathbf{P}^1$ bundles over $\mathbf{P}^1$ (ruled surfaces), the curve with self-intersection 0 is a fiber and the curve with self-intersection n is a section (see Šafarevič [9]). Thus there is an automorphism of $\mathbf{P}^1$ which acts on the compactification with graph $\overset{n-1}{\circ}\!-\!\!-\!\overset{0}{\circ}$ fixing the curve with self-intersection $n-1$ and permuting an arbitrary pair of points in the curve with self-intersection 0, as long as these points are distinct from the point of intersection of these two curves. Thus the compactifications with graph $\overset{n}{\circ}\!-\!\!-\!\overset{0}{\circ}$ are isomorphic in the required way.

Next we consider compactifications with graph $\overset{0}{\circ}\!-\!\overset{0}{\circ}\!-\!\overset{-1}{\circ}$. They are obtained via the sequence (5).

$$(5)\qquad \overset{0}{\circ}\!-\!\overset{0}{\circ} \longrightarrow \overset{0}{\circ}\!-\!\overset{-1}{\circ}\!-\!\overset{-1}{\circ} \longrightarrow \overset{-1}{\circ}\!-\!\overset{-1}{\circ}\!-\!\overset{-1}{\circ}\!-\!\overset{-1}{\circ} \longrightarrow \overset{0}{\circ}\!-\!\overset{0}{\circ}\!-\!\overset{-1}{\circ}\,.$$

Since we can find an automorphism of $\mathbf{P}^1\times\mathbf{P}^1$ preserving each factor, fixing $(0,0)$, permuting an arbitrary pair of points of $\mathbf{P}^1\times\{0\}-\{(,0 0)\}$, and also permuting an arbitrary pair of points of $\{0\}\times\mathbf{P}^1-\{(0,0)\}$, we see that any two compactifications with graph $\overset{0}{\circ}\!-\!\overset{0}{\circ}\!-\!\overset{-1}{\circ}$ are isomorphic in the required way.

Consider now the sequence

$$(6)\qquad \overset{n}{\circ}\!-\!\overset{0}{\circ}\!-\!\overset{-n-1}{\circ} \longrightarrow \overset{n}{\circ}\!-\!\overset{-1}{\circ}\!-\!\overset{-1}{\circ}\!-\!\overset{-n-2}{\circ} \longrightarrow \overset{n+1}{\circ}\!-\!\overset{0}{\circ}\!-\!\overset{-n-2,}{\circ}\quad n\geqq 0.$$

We see by induction that any two compactifications with graph $\overset{k}{\circ}\!-\!\overset{0}{\circ}\!-\!\overset{-k-1}{\circ}$, $k\geqq 0$ are isomorphic in the required way.

Next consider the sequence

$$(7)\qquad \overset{n}{\circ}\!-\!\!-\!\overset{0}{\circ} \longrightarrow \overset{-1}{\circ}\!-\!\overset{n-1}{\circ}\!-\!\overset{0}{\circ} \longrightarrow \overset{-2}{\circ}\!-\!\overset{-1}{\circ}\!-\!\overset{n-2}{\circ}\!-\!\overset{0}{\circ} \to$$

$$\cdots \to \underbrace{\overset{-2}{\circ}\!-\!\overset{-2}{\circ}\!-\cdots-\!\overset{-2}{\circ}}_{n-1\ \text{terms}}\!-\!\overset{-1}{\circ}\!-\!\overset{0}{\circ}\!-\!\overset{0}{\circ} \to \cdots$$

$$\to \underbrace{\overset{-2}{\circ}\!-\!\overset{-2}{\circ}\!-\cdots-\!\overset{-2}{\circ}}_{n-1\ \text{terms}}\!-\!\overset{-k-1}{\circ}\!-\!\overset{0}{\circ}\!-\!\overset{k}{\circ}\,.$$

Considering the method of constructing compactifications M with graph $\overset{n}{\circ}\!-\!\!-\!\overset{0}{\circ}$ we see that we can find an automorphism of M permuting any two points ($\neq 0$) on the curve with self-intersection n and fixing the curve with self-intersection 0. Thus the sequence described in (7) uniquely determines the compactifications with graph

$$\begin{array}{cccccc} -2 & -2 & & -2 & -k-1 & 0 & k \\ \circ\!-\!-\!-\!-\!-\!\circ & \!-\!-\cdots\!-\!- & \!\circ & \!-\!-\!-\!-\!\circ & \!-\!-\!-\!-\!-\!\circ & \!-\!-\!-\!\circ \end{array}$$

where k can be any integer.

Considering the induction proof of Theorem 9 and the facts just proved we see that all minimal normal compactification of $\mathbf{C}^2$ are uniquely determined by their graphs. Q.E.D.

We append here a list of open problems.

Let M be compact, complex, of dimension n. Let $A \subset M$, $A = \cup \Gamma_i$, each Γ_i a non-singular hypersurface and the Γ_i intersect normally. Suppose $M - A = \mathbf{C}^n$.

1. Is $\Gamma_i = \mathbf{P}^{n-1}$ for each i?
2. Is M algebraic?
3. Is M birationally equivalent to $\mathbf{P}^n$?
4. In case $M - A$ is some simple affine variety, what can one say about M, A?
5. What invariants should one use to classify compactifications of $\mathbf{C}^n$?

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